

The Color-Superconducting 't Hooft Interaction

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We consider the effect of a six-fermion interaction of the 't Hooft form in the quark-quark channel on the ground state of matter at finite density. The coupling constant for this new term is varied within the limits suggested by naturalness. The flavor-mixing effects of the additional term destabilize the color-flavor-locked (CFL) and, to a lesser extent, the two-flavor color superconducting (2SC) phases of quark matter, especially for positive values of the coupling. For some values of the coupling, the critical density for CFL phase is nearly larger than the maximum density in the neutron stars. We comment on the implications for neutron star evolution.

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HISTORY AND MOTIVATION

The study of dense (baryon chemical potential ~ 1.5 GeV) matter has been recently revolutionized by the observation that dense quark matter exhibits color-superconductivity and that the gaps may be of order 100 MeV [1, 2, 3]. Gaps of this magnitude are large enough to have significant implications for neutron star structure [4], proto-neutron star evolution [5], and neutron star cooling [6].

Since directly utilizing QCD at the relevant densities ($\mu_{\text{baryon}} \sim 1$ GeV) is so far impossible, the use of effective theories like the Nambu–Jona-Lasinio (NJL) model is common in the study of dense quark matter. In the NJL model, the high-energy degrees of freedom (the gluons) are integrated out and we restrict ourselves to working at energy scales less than the momentum cutoff Λ [7, 8]

$$\mathcal{L}_{\text{eff}} = \sum_n \frac{c_n}{\Lambda^{\dim(\mathcal{O}_n)-4}} \mathcal{O}_n \quad (1)$$

where \mathcal{O}_n are operators, $\dim(\mathcal{O}_n)$ is the dimension of the operator, and c_n are dimensionless coupling constants. Because it is impossible to create a model with the same symmetries as QCD with four-fermion operators alone the so-called 't Hooft term [9]

$$\mathcal{L} = \mathcal{L}_{\text{kinetic}} + \mathcal{L}_{\text{four-fermion}} + \mathcal{L}_{\text{'tHooft}} \quad (2)$$

$$\mathcal{L}_{\text{'tHooft}} \sim \det_f [\bar{q}(1 + \gamma_5)q] + \det_f [\bar{q}(1 - \gamma_5)q]$$

is added where q is a quark spinor and \det_f is a determinant in flavor space. This term, like QCD, respects chiral symmetry but breaks the $U(1)_A$ symmetry. The use of this Lagrangian is the standard approach which has been used to describe dense matter.

Unfortunately, when employed to study quark superconductivity, this standard approach does not employ a manifestly consistent truncation scheme; the quark–anti-quark interaction is treated at the six-fermion level, but the quark-quark interaction is only treated at the four-fermion level. There is no reason to rule out a term of the form (omitting color for clarity)

$$\mathcal{L}_{\text{CS6}} \sim K_{\text{DIQ}} \epsilon_{ijm} \epsilon_{k\ell n} (\bar{q}_i \gamma_5 q_j^C) (\bar{q}_k^C \gamma_5 q_\ell) (\bar{q}_m q_n) \quad (3)$$

which has the same symmetries as $\mathcal{L}_{\text{'tHooft}}$. This term may have a significant impact on the nature of dense matter [3, 10, 11]. While the effect of the dynamically generated quark masses on the superconducting gaps has been studied [12], Eq. 3 implies a modification to the quark masses due to the presence of the gap.

In this article, we study the effect of \mathcal{L}_{CS6} on the quark masses and on the phase structure of dense matter. We show that sufficiently positive values of K_{DIQ} increase the quark masses and thus favor less-gapped phases, while sufficiently negative values of K_{DIQ} split the magnitude of the up-down and light-strange gaps. The phase structure of dense matter thus depends critically on the sign and magnitude of this unknown parameter.

THE MODEL LAGRANGIAN

The Lagrangian is [12, 13, 14]

$$\begin{aligned} \mathcal{L} = & \bar{q}_{i\alpha} (i\partial\!\!\!/ - m_{0,ij} - \mu_{j\beta}^{i\alpha} \gamma_0) q_{j\beta} + G \sum_{a=0}^8 (\bar{q} \lambda_f^a q)^2 \\ & + G_{\text{DIQ}} \epsilon^{ijm} \epsilon^{k\ell n} \epsilon^{\alpha\beta\epsilon} \epsilon^{\gamma\delta\epsilon} \\ & \times (\bar{q}_{i\alpha} i\gamma^5 q_{j\beta}^C) (\bar{q}_{k\gamma}^C i\gamma^5 q_{\ell\delta}) \\ & - K [\det_f \bar{q} (1 - i\gamma^5) q + \det_f \bar{q} (1 + i\gamma^5) q] \\ & + K_{\text{DIQ}} \epsilon^{ijm} \epsilon^{k\ell n} \epsilon^{\alpha\beta\epsilon} \epsilon^{\gamma\delta\eta} \\ & \times (\bar{q}_{i\alpha} i\gamma_5 q_{j\beta}^C) (\bar{q}_{k\gamma}^C i\gamma_5 q_{\ell\delta}) (\bar{q}_m \epsilon q_n) \end{aligned} \quad (4)$$

where flavor is represented by Latin indices, color is represented by Greek indices, and the charge conjugate Dirac spinors are defined by $(C\gamma^\mu C = \gamma^{\mu T}$ and $C^T = -C)$

$$q^C \equiv C\bar{q}^T \quad \text{and} \quad \bar{q}^C = q^T C. \quad (5)$$

The four-fermion coupling in the quark–anti-quark channel is denoted G , the four-fermion coupling in the quark-quark channel is denoted G_{DIQ} , and $m_{0,ij}$ is the constant current quark mass matrix which is diagonal in flavor.

We utilize the ansätze

$$\begin{aligned}
\bar{q}_1 q_2 \bar{q}_3 q_4 &\rightarrow \langle \bar{q}_1 q_2 \rangle \bar{q}_3 q_4 + \bar{q}_1 q_2 \langle \bar{q}_3 q_4 \rangle - \langle \bar{q}_1 q_2 \rangle \langle \bar{q}_3 q_4 \rangle \\
\bar{q}_1 q_2 \bar{q}_3 q_4 \bar{q}_5 q_6 &\rightarrow \bar{q}_1 q_2 \langle \bar{q}_3 q_4 \rangle \langle \bar{q}_5 q_6 \rangle + \langle \bar{q}_1 q_2 \rangle \langle \bar{q}_3 q_4 \rangle \bar{q}_5 q_6 \\
&\quad + \langle \bar{q}_1 q_2 \rangle \bar{q}_3 q_4 \langle \bar{q}_5 q_6 \rangle \\
&\quad - 2 \langle \bar{q}_1 q_2 \rangle \langle \bar{q}_3 q_4 \rangle \langle \bar{q}_5 q_6 \rangle
\end{aligned} \tag{6}$$

to obtain the mean-field approximation [15]. This procedure retains only the lowest order terms in the $1/N_c$ expansion [16]. Color neutrality is ensured using the procedure from Ref. [12]. In the mean-field approximation, the Lagrangian is only quadratic in the fermion fields, and the thermodynamical potential can be obtained from the inverse propagator in the standard way [17]. The momentum integrals in the gap equations are divergent, and are regulated by a three-momentum cutoff denoted by Λ . The inverse propagator is numerically diagonalized for each abscissa of the momentum integration to obtain the thermodynamical potential.

For simplicity, we sometimes use the notation $\Delta_k \sim \epsilon^{ijk} \langle q_i^C \gamma^5 q_j \rangle$, so that gaps are denoted with the flavor of quark that is *not* involved in the pairing e.g. Δ_{ud} is denoted by Δ_s . Other than $\langle \bar{q}_i q_i \rangle$ and Δ_i , we assume that all other condensates vanish. This includes the pseudoscalar condensates which are likely present in dense matter and naturally accompany the Goldstone bosons [18]. This (non-trivial) complication will be left to later work.

The effects of \mathcal{L}_{CS6} on the thermodynamical potential can be summarized in three modifications from the standard approach where $K_{\text{DIQ}} = 0$. These changes are that the values of the gap in the inverse propagator are modified

$$\Delta_i \rightarrow \Delta_i \left(1 + \frac{K_{\text{DIQ}}}{N_c G_{\text{DIQ}}} \langle \bar{q}_i q_i \rangle \right), \tag{7}$$

a new effective mass term (which includes contributions which are not diagonal in flavor) appears

$$\frac{K_{\text{DIQ}}}{4G_{\text{DIQ}}^2} \bar{q}_i \Delta_i \Delta_j q_j, \tag{8}$$

which modifies the dynamical mass

$$m_i^* = m_{i,0} - 4G \langle \bar{q} q \rangle_i + K |\epsilon_{ijk}| \langle \bar{q} q \rangle_j \langle \bar{q} q \rangle_k + \frac{K_{\text{DIQ}}}{4G_{\text{DIQ}}^2} \Delta_i^2 \tag{9}$$

and that there is a new contribution to the vacuum energy

$$\Omega_{K_{\text{DIQ}}} = \frac{K_{\text{DIQ}}}{2G_{\text{DIQ}}^2} \sum_i \Delta_i^2 \langle \bar{q}_i q_i \rangle. \tag{10}$$

Note that Eq. 8 means that the quark masses are density-dependent if $K_{\text{DIQ}} \neq 0$ even when the chiral condensates $\langle \bar{q} q \rangle$ vanish. As has been suggested [19], this term

generates a dynamical quark mass entirely distinct from the typical mechanism of spontaneous chiral symmetry breaking. The quarks obtain a dynamical mass even when the quark condensate is taken to be zero.

One may use Fierz transformations to calculate the coefficient K_{DIQ} from the quark-anti-quark form of the 't Hooft interaction. One can view this in the following way: For each prospective new term, e.g. the term $\bar{u} \gamma^5 d_c \bar{d}_c \gamma^5 u \bar{s} s$, there are six Fierz transformations (for six fermions this is a 35×35 matrix instead of the usual 5×5 matrix for four fermions) that give this term when applied to the six sets (in flavor space) of terms in the 't Hooft interaction. One such transformation is

$$\begin{aligned}
&\bar{q}_i q_j \bar{q}_k q_\ell \bar{q}_m q_n + \bar{q}_i \gamma^5 q_j \bar{q}_k \gamma^5 q_\ell \bar{q}_m q_n \\
&+ \bar{q}_i q_j \bar{q}_k \gamma^5 q_\ell \bar{q}_m \gamma^5 q_n + \bar{q}_i \gamma^5 q_j \bar{q}_k q_\ell \bar{q}_m \gamma^5 q_n \\
&= \frac{1}{2} (\bar{q}_i \gamma^5 q_k^C \bar{q}_j^C \gamma^5 q_\ell \bar{q}_m q_n + \bar{q}_i q_k^C \bar{q}_j^C \gamma^5 q_\ell \bar{q}_m \gamma^5 q_n \\
&\quad + \bar{q}_i q_k^C \gamma^5 \bar{q}_j^C q_\ell \bar{q}_m \gamma^5 q_n + \bar{q}_i q_k^C \bar{q}_j^C q_\ell \bar{q}_m q_n) \\
&\quad - \frac{1}{4} (\bar{q}_i \gamma^5 \sigma^{\mu\nu} q_k^C \bar{q}_j^C \gamma^5 \sigma_{\mu\nu} q_\ell \bar{q}_m q_n \\
&\quad + \bar{q}_i \gamma^5 \sigma^{\mu\nu} q_k^C \bar{q}_j^C \sigma_{\mu\nu} q_\ell \bar{q}_m \gamma^5 q_n)
\end{aligned} \tag{11}$$

When these transformations are combined to give the coefficient K_{DIQ} , the result is zero (see the Appendix). Although terms with different Dirac structure do survive, we do not include these terms since we do not expect the corresponding condensates, e.g. $\langle \bar{q} \sigma_{\mu\nu} q \rangle$ to be non-zero. This procedure is not the only possible approach for deriving the mean-field Lagrangian (one could also enumerate all possible Wick contractions). Because alternate approaches and/or using terms of higher order in $1/N_c$ may modify this result, we cannot conclude necessarily that K_{DIQ} must be zero. Also, terms like Eq. 2 with a Dirac structure $(\bar{q} q)(\bar{q} \sigma_{\mu\nu} q)(\bar{q} \sigma^{\mu\nu} q)$ [20] and their corresponding terms in the quark-quark channel may play a role. We leave these considerations to future work.

Our Lagrangian is free to contain any terms which follow the symmetries of the underlying theory. We expect that the coefficient of this term will be “natural”. When the coefficients are expressed in terms of the underlying length scales (in our case, the momentum cutoff Λ), the coefficients should all be of similar magnitude. We allow the coefficient K_{DIQ} to vary, between the values $-K$ and K , which we view to a modest variation as suggested by the constraints of naturalness. A larger variation in K_{DIQ} is not necessarily excluded. We use the values of Λ and the current quark masses from Ref. [21], where they are fixed by matching the pion, kaon, and η' masses in vacuum as well as the pion decay constant. We choose to fix $G_{\text{DIQ}} \Lambda^2 = 1.61$ to be large enough so that the maximum value of the gaps (when including the quark dynamical mass) as a function of density when $K_{\text{DIQ}} = 0$ is about 80 MeV, close to the value of about 100 MeV predicted by calculations in perturbative QCD. If the dynamically-generated quark mass is assumed to be zero

and the mass-gap equations are ignored, then the maximum value of the gap predicted by this model is about 120 MeV. We leave G_{DIQ} fixed when varying K_{DIQ} . One could also allow the coefficient, G_{DIQ} as a function to vary as a function of K_{DIQ} by instead ensuring that the maximum value of one of the three gaps at high densities is constant. We have checked that this alternative does not change our conclusions significantly.

RESULTS

Obtaining analytical results is difficult, due to the flavor mixing mass terms from Eq. 8 which make it difficult to directly reduce the inverse propagator (a 36×36 matrix) into a block-diagonal form. It is possible, with High-Density Effective Theory [22], to simplify the Dirac structure, but this would likely result in a 9×9 inverse propagator which is also difficult. It is also possible to restore the usual form of the propagator encountered in studies where $K_{\text{DIQ}} = 0$ by ignoring the terms in Eq. 8 where $i \neq j$. In this case, the inverse propagator is worked out in detail in Ref. [23].

It is possible to see qualitatively, what the effect of adding a term with $K_{\text{DIQ}} \neq 0$ might be from Eq. 7 above. When $K_{\text{DIQ}} > 0$, we expect the gaps decrease as the quark condensate increases, and thus the gap should decrease with increasing mass. However, from Eq. 9 we expect the opposite and we find that it is this effect that dominates the description of the strange quark mass and Δ_{ud} . Further complicating the analysis, Eq. 8 indicates that an increase in Δ_{us} and Δ_{ds} will tend to split the mass of the up and down quark, thus possibly weakening Δ_{ud} .

We study charge- and color-neutral, beta-equilibrated, bulk matter at fixed baryon density and a fixed temperature of 10 MeV. We operate at a small but finite temperature in order to alleviate the numerical difficulties of discontinuities in the momentum integral present in the thermodynamical potential. The zero-temperature results will not deviate significantly from our results. We include non-interacting electrons, but we do not include neutrinos. The addition of neutrinos would further split the approximate flavor symmetry between the up and down quarks. Our results will faithfully describe matter in the center of a neutron star containing quarks a minute or later after formation [24, 25].

We note that the effect of K_{DIQ} is small when the quark condensates $\langle \bar{q}q \rangle$ are taken to be zero. When this is assumed to be true, then the modifications from Eqs. 7 and 8 have no effect, and the gaps are nearly independent of K_{DIQ} . We remove this assumption and solve the mass gap equations for the quark condensates in the following.

Figure 1 presents the masses and gaps in the CFL phase at fixed density and temperature as a function of K_{DIQ} . Both the quark masses and Δ_{ud} increase as

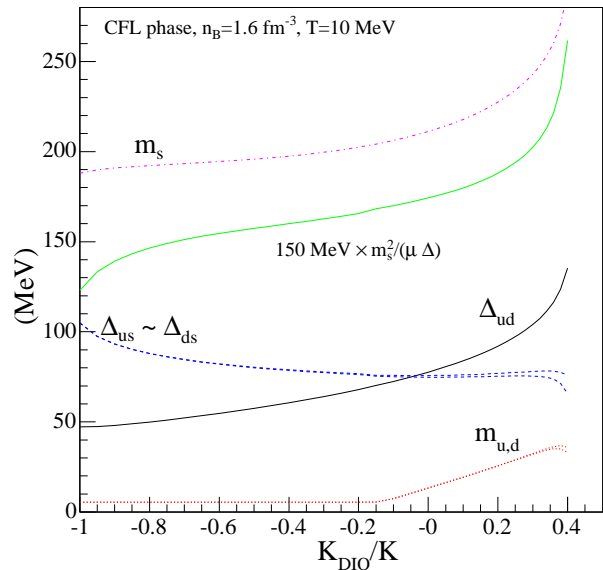


FIG. 1: Quark masses (dotted lines for u and d, and dashed-dotted line for s) and gaps (solid line for Δ_{ud} and dashed line for the other two gaps) as a function of K_{DIQ} in the CFL phase at $n_B = 1.6 \text{ fm}^{-3}$.

K_{DIQ} increases. The effect from Eq. 8 causes the Δ_{us} and Δ_{ds} gap to decrease when Δ_{ud} increases. At sufficiently large values of the coupling, the strong increase of the strange quark mass destabilizes the CFL phase. For $K_{\text{DIQ}} > 0.4$, the gap equations have no solution. If the coupling was verified by some other means to be larger than this critical value, then the CFL phase could not be present at this density. As $K_{\text{DIQ}}/K \rightarrow -1$, the effects on the masses and gaps tend to be less extreme. The most significant effect is the increasing split between the values of the light-strange gaps, Δ_{us} and Δ_{ds} , and the light-quark gap Δ_{ud} . One might expect the dependence of the gaps on K_{DIQ} would change the phase structure of matter by shifting the energy density. However, we find that this is not the case and that the energy density is relatively constant as a function of K_{DIQ} . Note also that the strange quark mass can change by as much as 50% for different values of K_{DIQ} .

Also plotted in Fig. 1 is the parameter $m_s^2/(\mu\Delta)$ where μ and Δ in this context are computed by averaging the quark chemical potentials and gaps over all flavors and colors. This parameter has been demonstrated to be the relevant dimensionless quantity which dictates the phase content of quark matter at high density [26]. Values of $m_s^2/(\mu\Delta)$ larger than 2 suggest a transition to a gapless CFL phase [27], while values larger than 4 suggest a transition to the 2SC phase. The presence of the gapless 2SC phase [28] is also probable when K_{DIQ} becomes positive, since the ud gap is becoming stronger and the

light-strange gaps are weakening. The value of $m_s^2/(\mu\Delta)$ is also important in dictating the number and type of Goldstone bosons present in the CFL phase. This parameter is quite flat for small variations in K_{DIQ} , but increases or decreases strongly when the absolute magnitude of K_{DIQ} is sufficiently large. When $K_{\text{DIQ}}/K \sim 0.4$, the value of $m_s^2/(\mu\Delta)$ is nearly 2, indicating the disappearance of the CFL phase.

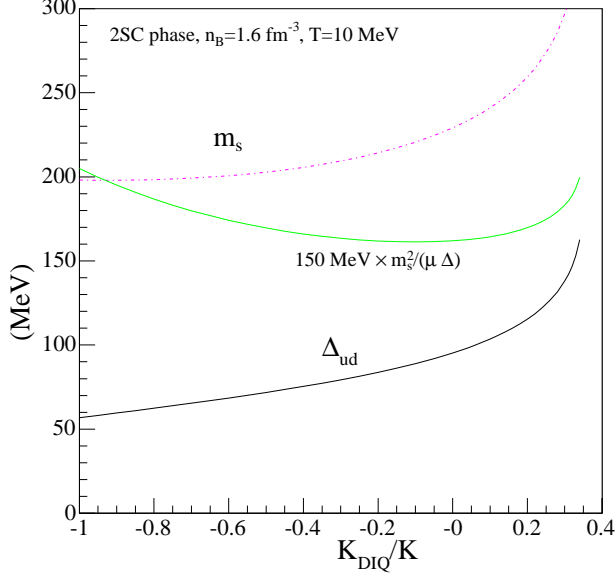


FIG. 2: Strange quark mass (dashed-dotted line) and the gap (solid line) as a function of K_{DIQ} in the 2SC phase at $n_B = 1.6 \text{ fm}^{-3}$.

The results for the 2SC phase, where $\Delta_{us} = \Delta_{ds} = 0$, at the same density and temperature are shown in Figure 2. The strange quark mass and gap increase strongly for increasing K_{DIQ} as in the CFL phase, leading to a critical value above which the gap equations have no solution. At this density, the 2SC phase is also not present in matter for $K_{\text{DIQ}} > 0.4$. The similarity of this critical value of the coupling to the CFL phase in Fig. 1 is a result of the fact that the light-strange gaps and light-quark masses are comparatively small and thus do not strongly perturb m_s and Δ_{ud} . The parameter $m_s^2/(\mu\Delta)$ is slightly modified from the CFL case and indicates that the 2SC phase is also likely to be unstable for large negative values of K_{DIQ}/K .

For comparison, Fig. 3 presents the quark masses and gaps in the CFL phase at a lower density, 1.2 fm^{-3} . The results are not much different from Fig. 1. The quark gaps have decreased slightly and the quark masses are slightly larger. The major distinction is that the critical value of K_{DIQ} above which the CFL phase does not appear has been lowered from 0.4 to less than 0.3.

The implication of the shift in the critical value of K_{DIQ} is that the critical density for the onset of the

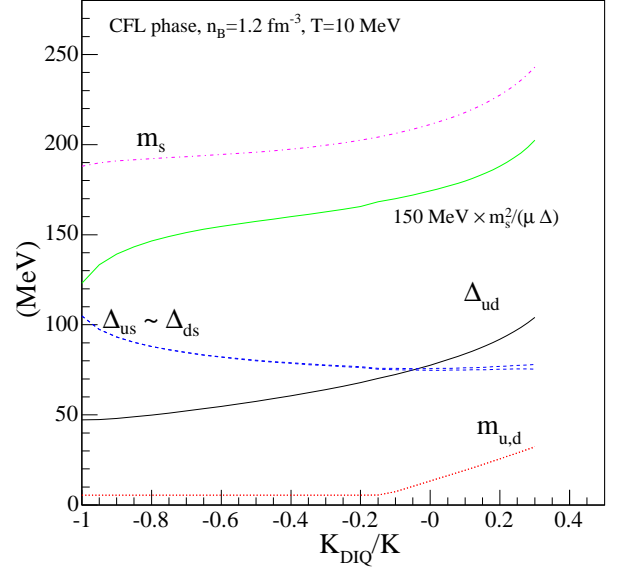


FIG. 3: The quark masses and gaps in the CFL phase at $n_B = 1.2 \text{ fm}^{-3}$. See Fig. 1.

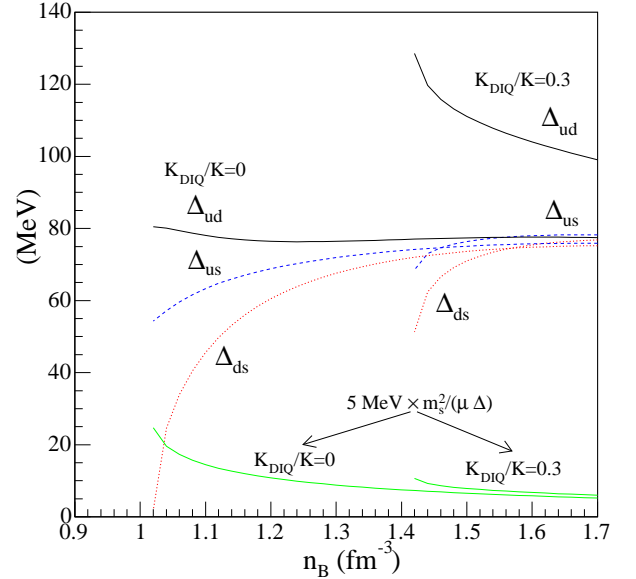


FIG. 4: The gaps as a function of baryon density in the CFL phase for two values of the coupling K_{DIQ} .

CFL phase is drastically affected by a positive value of K_{DIQ} . This is demonstrated in Fig. 4, where the smallest quark gap vanishes at about 1.0 fm^{-3} when $K_{\text{DIQ}} = 0$, and at about 1.42 fm^{-3} when $K_{\text{DIQ}}/K = 0.3$. On the other hand, because the gaps are not as strongly modified when K_{DIQ} is negative, the critical density when $K_{\text{DIQ}}/K = -0.5$ is almost unchanged, moving down only

to 0.96 fm^{-3} . In this figure, the Δ_{ds} gap does not vanish completely to zero because the solution of the gap equations becomes difficult when the gaps are small. Note again that an increase the parameter $m_s^2/(\mu\Delta)$ indicates, to some extent, the disappearance of the CFL phase as the density decreases.

DISCUSSION - NEUTRON STARS WITH $K_{\text{DIQ}} \neq 0$

These results may have several implications for neutron star structure and evolution.

Neutron-star masses and radii: We have solved the Tolman-Oppenheimer-Volkov equations for neutron star structure under the assumption that there is no mixed phase (i.e. the surface tension is very large so that mixed phases are suppressed). The results for $K_{\text{DIQ}} = 0$ and $K_{\text{DIQ}}/K = 0.3$ are given in Fig. 5. We find that neutron star masses and radii are not very sensitive to K_{DIQ} for the model that we have chosen. As mentioned above, a positive value of K_{DIQ} tends to increase the critical density for the appearance of the CFL phase, thus stiffening the equation of state and slightly increasing the maximum mass from 1.83 to $1.9 M_\odot$. The small magnitude of this effect is partially due to the fact that, for $K_{\text{DIQ}} = 0$, the critical density for the appearance of quark matter is 1.0 fm^{-3} , while the central density of the maximum mass neutron star is only 1.45 fm^{-3} . There is not much quark matter to begin with, so therefore the effect of K_{DIQ} is limited. In regards to the phase content of matter in the neutron star, the larger value of K_{DIQ} nearly pushes the CFL phase out of the neutron star entirely, and the center of the neutron star is dominated by the 2SC phase instead. These effects will be larger if the diquark coupling is increased and may be modified by the presence of a mixed phase. Recent calculations of the surface tension suggest that it is small, and thus a mixed phase may be present [29]. It would be interesting to examine the effect of K_{DIQ} on this surface tension.

Neutron-star cooling: Neutron stars are sensitive to the difference between the CFL and 2SC phases since the former is likely to contain no [30] (or very few) electrons. Neutron star cooling is affected by the presence or absence of electrons because the specific heat of matter is dominated by the electrons when they are present. In addition, the specific heat contribution from the light quarks is proportional to $\exp(-\Delta_{\ell s}/T)$, where $\ell = u$ or d which is exponentially small in the CFL phase and of order unity in the 2SC phase. Also, the splitting of the gaps at $K_{\text{DIQ}}/K = -1$, will enforce two critical temperatures for quark matter in the CFL phase. The so-called “gapless” phases are dependent upon the strange quark mass which is strongly affected by K_{DIQ} , especially when it is greater than zero. These gapless phases, because of the nature of the quark dispersion relations, have unusual

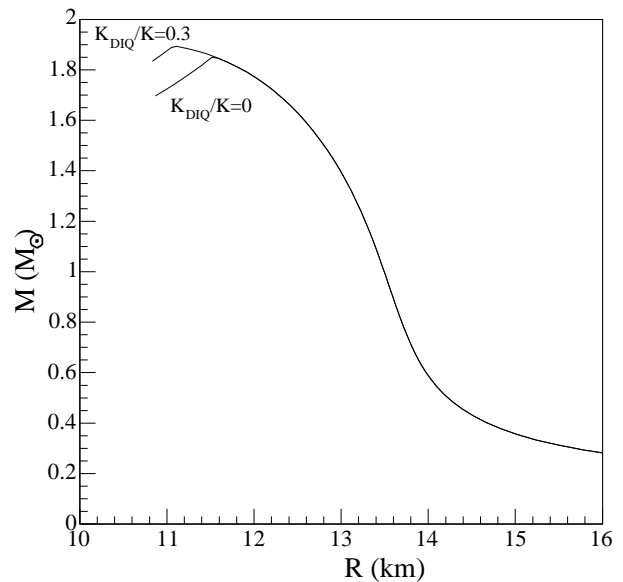


FIG. 5: The mass versus radius for neutron stars with two values of the coupling K_{DIQ} .

transport properties that also have implications for neutron star cooling [27, 28, 31].

Proto-neutron star evolution: Ref. [5] pointed out that the proto-neutron star neutrino signal may be increased noticeably by the enhanced cooling that is present when the core temperature falls below the critical temperature. The presence of K_{DIQ} implies that this enhanced cooling may occur in (at least) two separate stages, as the critical temperatures corresponding to the Δ_{ud} and the light-strange gaps are surpassed.

To the extent that the presence of a large ($\sim 100 \text{ MeV}$) gap affects observations of neutron stars, the presence of a color-superconducting six-fermion interaction also has an impact on neutron star observations. It would be interesting to compare these results with those from gapless CFL phases. Because of the uncertainty in the value of the K_{DIQ} coupling (in addition to the other uncertainties already present), it will be difficult to settle the questions of the nature of dense matter and the question of the properties of neutron stars containing deconfined quark matter until theoretical progress in QCD or astrophysical observations can more completely settle the issue.

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Appendix A - Fierz Transformations

Traditionally, the Fierz transformation is defined as the matrix, C , which obeys the relation

$$\mathcal{F} [\bar{\psi}_i \Gamma_a \psi_j \bar{\psi}_k \Gamma_{kl}^a \psi_\ell] = \sum_b C_{a,b} \bar{\psi}_i \Gamma_b \psi_\ell \bar{\psi}_k \Gamma_{kj}^b \quad (12)$$

where $\Gamma_i \in [\mathbb{1}, \gamma^5, \gamma_\mu, \gamma^5 \gamma_\mu, \sigma_{\mu\nu}]$ and $\Gamma^i \in [\mathbb{1}, \gamma^5, \gamma^\mu, \gamma^5 \gamma^\mu, \sigma^{\mu\nu}]$ for $i = a, b$. Because the Fierz transformation is nothing other than a set of equalities, the two four-fermion interactions on both sides are necessarily equivalent. However, in the mean-field approximation, these two forms lead to different thermodynamic potentials. For simplicity, the transformation above will be denoted $\mathcal{F}(ijkl \rightarrow ilkj)$.

One may also perform a four-fermion Fierz transformation in the quark-quark channel, namely, $\mathcal{F}(ijkl \rightarrow ikjl)$ (see the review in Ref. [32]). This transformation operates over a different “basis” of combinations of Dirac matrices

$$\mathcal{F} [\bar{\psi}_i \Gamma_a \psi_j \bar{\psi}_k \Gamma_{kl}^a \psi_\ell] = \sum_b C'_{a,b} \bar{\psi}_i \Gamma'_b \psi_\ell \bar{\psi}_k \Gamma_{kj}^C \Gamma_{jl}^b \psi_\ell. \quad (13)$$

where $\Gamma'_i \in [\gamma^5, \mathbb{1}, \gamma^5 \gamma_\mu, \gamma_\mu, \sigma_{\mu\nu}]$ and $\Gamma^{i,\prime} \in [\gamma^5, \mathbb{1}, \gamma^5 \gamma^\mu, \gamma^\mu, \sigma^{\mu\nu}]$ (remember that $\bar{\psi}^C \gamma^5 \psi$ is a Lorentz scalar). We can simplify the notation for the basis by using the notation of a direct product: $\gamma^5 \otimes \gamma^5, \mathbb{1} \otimes \mathbb{1}, \gamma^5 \gamma_\mu \otimes \gamma^5 \gamma_\mu, \gamma_\mu \otimes \gamma_\mu, \sigma_{\mu\nu} \otimes \sigma^{\mu\nu}$, or more simply, SS, PP, VV, AA, TT.

In a similar notation the 35-element basis for the six-fermion Fierz transformations is

$$\begin{aligned} & \text{SSS, SPP, PSP, PPS, SVV, VSV, VVS,} \\ & \text{SAA, ASA, AAS, STT, TST, TTS,} \\ & \text{PVA, PAV, VPA, VAP, APV, AVP,} \\ & \text{TVV, VTV, VVT, TAA, ATA, AAT, TTT,} \\ & \text{VAQ, VQA, AVQ, AQV, QVA, QAV,} \\ & \text{PTQ, TPQ, QTP} \end{aligned} \quad (14)$$

where Q denotes a “pseudo-tensor” combination, $\gamma^5 \sigma^{\mu\nu}$. (The Q terms are used as an alternative to the formulation in terms of objects of the form $\varepsilon_{\kappa\lambda\mu\nu} \sigma^{\mu\nu}$ [33].) All other combinations can be written as a linear combination of these 35 basis elements. In order to distinguish 4- and 6-fermion transformations, we will use a subscript, i.e. $\mathcal{F}_4(ijklmn \rightarrow ilkjmn)$ is really a four-fermion transformation since the fields with indices m and n are not participating in the transformation.

The Dirac scalar terms in the 't Hooft interaction are

$$\begin{aligned} & \bar{u}u\bar{d}d\bar{s}s + \bar{u}s\bar{d}u\bar{s}d + \bar{u}d\bar{s}s\bar{u} \\ & - \bar{u}d\bar{d}u\bar{s}s - \bar{u}s\bar{d}d\bar{s}u - \bar{u}u\bar{d}s\bar{s}d \end{aligned} \quad (15)$$

plus the corresponding terms created by adding an even number of γ^5 matrices.

As a demonstration, we examine the coefficient of the term $\bar{u}\gamma^5 d^C \bar{u}^C \gamma^5 d\bar{s}s$. The various contributions to this coefficient are

$$\begin{aligned} & +\mathcal{F}_4(ijklmn \rightarrow ikjlmn) [\bar{u}u\bar{d}d\bar{s}s + \bar{u}u\bar{d}\gamma^5 d\bar{s}\gamma^5 s \\ & \quad + \bar{u}\gamma^5 u\bar{d}d\bar{s}\gamma^5 s + \bar{u}\gamma^5 u\bar{d}\gamma^5 d\bar{s}s] \\ & +\mathcal{F}_6(ijklmn \rightarrow iklnmj) [\bar{u}s\bar{d}u\bar{s}d + \bar{u}s\bar{d}\gamma^5 u\bar{s}\gamma^5 d \\ & \quad + \bar{u}\gamma^5 s\bar{d}u\bar{s}\gamma^5 d + \bar{u}\gamma^5 s\bar{d}\gamma^5 u\bar{s}d] \\ & +\mathcal{F}_6(ijklmn \rightarrow iknjml) [\bar{u}d\bar{s}s\bar{u} + \bar{u}d\bar{d}\gamma^5 s\bar{s}\gamma^5 u \\ & \quad + \bar{u}\gamma^5 d\bar{s}s\bar{s}\gamma^5 u + \bar{u}\gamma^5 d\bar{d}\gamma^5 s\bar{s}u] \\ & -\mathcal{F}_4(ijklmn \rightarrow ikljmn) [\bar{u}d\bar{d}u\bar{s}s + \bar{u}d\bar{d}\gamma^5 u\bar{s}\gamma^5 s \\ & \quad + \bar{u}\gamma^5 d\bar{d}u\bar{s}\gamma^5 s + \bar{u}\gamma^5 d\bar{d}\gamma^5 u\bar{s}s] \\ & -\mathcal{F}_6(ijklmn \rightarrow iknlmj) [\bar{u}s\bar{d}d\bar{s}u + \bar{u}s\bar{d}\gamma^5 d\bar{s}\gamma^5 u \\ & \quad + \bar{u}\gamma^5 s\bar{d}d\bar{s}\gamma^5 u + \bar{u}\gamma^5 s\bar{d}\gamma^5 d\bar{s}u] \\ & -\mathcal{F}_6(ijklmn \rightarrow ikjnm) [\bar{u}u\bar{d}s\bar{s}d + \bar{u}u\bar{d}\gamma^5 s\bar{s}\gamma^5 d \\ & \quad + \bar{u}\gamma^5 u\bar{d}s\bar{s}\gamma^5 d + \bar{u}\gamma^5 u\bar{d}\gamma^5 s\bar{s}d] \end{aligned}$$

Note that result of the first of these six transformations is given as Eq. 11 in the text. The coefficients of the desired term, $\bar{u}\gamma^5 d^C \bar{u}^C \gamma^5 d\bar{s}s$, from each of the six transformations (together with an appropriate factor of -1 for odd fermionic permutations) are

$$\frac{1}{2}, -\frac{1}{4}, -\frac{1}{4}, -\frac{1}{2}, \frac{1}{4}, \frac{1}{4} \quad (16)$$

and the sum is zero.

Because the coefficient is zero, we need not consider the Fierz transformations in the SU(3) (color or flavor) spaces. However, since the result for six-fermion transformations in SU(3) is not present in the literature, we give the 15-element basis for computing the transformations (this enlarged basis from that presented in Ref. [34] is necessary to perform the transformations in the quark-quark channel)

$$\begin{aligned} & \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1}, \quad \lambda^a \otimes \lambda_a \otimes \mathbb{1}, \\ & \mathbb{1} \otimes \lambda^a \otimes \lambda_a, \quad \lambda_a \otimes \mathbb{1} \otimes \lambda^a, \\ & d_{abc} \lambda^a \otimes \lambda^b \otimes \lambda^c, \quad if_{abc} \lambda^a \otimes \lambda^b \otimes \lambda^c, \\ & \lambda(A)^a \otimes \lambda(A)_a \otimes \mathbb{1}, \quad \mathbb{1} \otimes \lambda(A)^a \otimes \lambda(A)_a, \\ & \lambda(A)_a \otimes \mathbb{1} \otimes \lambda(A)^a, \\ & \lambda(A)_a \lambda(S)_b \otimes \lambda(A)^a \otimes \lambda(S)^b, \\ & \lambda(A)_a \lambda(S)_b \otimes \lambda(S)^b \otimes \lambda(A)^a, \\ & \lambda(S)^b \otimes \lambda(A)_a \lambda(S)_b \otimes \lambda(A)^a, \\ & \lambda(A)^a \otimes \lambda(A)_a \lambda(S)_b \otimes \lambda(S)^b, \\ & \lambda(A)^a \otimes \lambda(S)^b \otimes \lambda(A)_a \lambda(S)_b, \\ & \lambda(S)^b \otimes \lambda(A)^a \otimes \lambda(A)_a \lambda(S)_b \end{aligned} \quad (17)$$

where the occurrence of λ indicates an implicit sum over all 8 SU(3) matrices, $\lambda(A)$ restricts the sum to only the

three anti-symmetric λ matrices, and the implicit sum for $\lambda(S)$ is over the six symmetric λ matrices. The full results are available from the author.

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